UNSTEADY FLOW OF VISCO-PLASTIC MATERIAL IN A CIRCULAR TUBE

(NEUSTANOVIVSHEESIA TECHENIE VIAZKO-PLASTICHESKOGO MATERIALA V KRUGLOI TRUBE)

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A.I. SAFRONCHIK
(Saratov)

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This paper presents the formulation and solution of the problem of unsteady flow of visco-plastic material in a circular tube for a variable pressure gradient. A method of solution of the axisymmetric "problem with unknown boundary" is developed for the equations of heat conduction. The law of change of the "kernel" of the flow with time is determined.

1. Formulation of the problem. We consider the flow of visco-plastic material in a circular tube of radius R under the action of a given pressure drop. We will assume the tube to be sufficiently long (in order to neglect the influence of the ends), the material of the tube to be infinitely rigid, and the visco-plastic material to be incompressible. We direct the Oz axis along the tube in the direction of motion and the axis Or along one of the radii. The equation of motion in cylindrical coordinates on the basis of the assumption of axial symmetry has the form

$$\rho \frac{\partial v}{\partial t} = \mu \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) + \frac{\tau_0}{r} + P(t) \qquad \left(P(t) = -\frac{\partial p}{\partial z} \right) \tag{1.1}$$

Here v is the axial velocity component, ρ the density, r the coefficient of viscosity, $P(t)$ the pressure drop on a unit length of tube, r_0 the limiting shear stress.

The given pressure drop will be assumed to be sufficiently large so that the corresponding shear stresses exceed the limiting shear stress. The entire region of flow consists of two parts: the region of actual visco-plastic flow ($r_0(t) < r < R$) in which the distribution of velocity is described by (1.1), and the region of the elastic "kernel" ($0 < r < r_0(t)$), moving as a rigid body. Let $r_0(t)$ denote the radius of the kernel which is an unknown function of time, subject to determination.

The critical condition will be taken in the form

$$v(r,0) = F(r) \quad \text{for} \quad r_0(0) < r < R \tag{1.2}$$

Here $r(0)$ is the initial radius of the kernel.

At the wall of the tube, as in the case of a viscous liquid, one will have the condition of attachment

$$v(r,t) = 0 \quad \text{for} \quad r = R \tag{1.3}$$

Since at the boundary of the "kernel" the stress is equal to the limiting shear stress

$$\frac{\partial v}{\partial r} = 0 \quad \text{for} \quad r = r_0(t) \tag{1.4}$$

The second condition on the boundary of the kernel we obtain by considering it as a body of variable mass, which changes with the variations of the area of its cross-section. Applying the law of conservation of momentum to the mass of unit length and taking into consideration that mass is added (taken away) without shock, we find

$$\rho \frac{dv_0(t)}{dt} = P(t) - \frac{2\tau_0}{r_0(t)} \tag{1.5}$$

Here $v_0(t)$ is the velocity of the kernel. Integrating (1.5) with respect to t, we will have

$$v_0(t) = v_0(0) + \frac{1}{\rho}\int_0^t P(\sigma)\,d\sigma - \frac{2\tau_0}{\rho}\int_0^t \frac{d\sigma}{r_0(\sigma)} \tag{1.6}$$

We introduce now non-dimensional time, radius x and velocity u by the formulas

$$y = \frac{\nu}{R^2}t, \qquad x = \frac{r}{R}, \qquad u = \frac{v}{V}$$

We will reduce the equation (1.1) and the boundary conditions to the non-dimensional form

$$\frac{\partial U}{\partial y} = \frac{\partial^2 U}{\partial x^2} + \frac{1}{x}\frac{\partial U}{\partial x} + \frac{S}{x} + P_*(y) \quad \text{for} \quad y > 0,\ \delta(y) < x < 1 \tag{1.7}$$

For the construction of the solution it is sufficient to assume the boundary condition fulfilled for the limit

$$\lim_{y \to +0} U(x,y) = \frac{F(Rx)}{V} = \Phi(x) \quad \text{for} \quad \delta_0 < x < 1 \tag{1.8}$$

$$\lim_{x \to 1-0} U(x,y) = 0, \qquad \lim_{x \to \delta(y)+0} \frac{\partial u}{\partial x} = 0 \qquad (y > 0) \tag{1.9}$$

$$\lim_{x\to\delta(y)+0} U(x, y) = \Phi(\delta_0) + \int_0^y \left[P_*(\sigma) - \frac{2S}{\delta(\sigma)}\right] d\sigma \qquad (y > 0) \tag{1.10}$$

Here $P_*(y)$ is the non-dimensional pressure jump per unit length of the tube, S is the St. Venant parameter, $\delta(y)$ the non-dimensional radius of the kernel

$$P_*(y) = \frac{R^2}{\mu V} P(t), \qquad S = \frac{\tau_0 R}{\mu V}, \qquad \delta(y) = \frac{r_0(t)}{R} \tag{1.11}$$

2. Construction of the solution. For the solution of the formulated problem with unknown movable boundary we will use a method which has been proposed by Kolodner for linear problems [1]. We will seek the solution of the problem (1.7)-(1.10) in the form

$$U(x, y) = -Sx + \int_0^y P_*(\sigma)\, d\sigma + K(x, y) + \lambda(x, y) \tag{2.1}$$

The function $K(x, y)$ we will choose such that it satisfies the equation

$$\frac{\partial K}{\partial y} = \frac{\partial^2 K}{\partial x^2} + \frac{1}{x}\frac{\partial K}{\partial x} \tag{2.2}$$

and the condition

$$\lim_{y\to+0} K(x, y) = Sx + \Phi(x) \qquad \text{for } \delta_0 < x < 1 \tag{2.3}$$

In the capacity of such a function we may take, for example,

$$K(x, y) = \frac{1}{2y}\int_{\delta_0}^1 \xi\,[S\xi + \Phi(\xi)]\, I_0\left(\frac{x\xi}{2y}\right) \exp\frac{-(x^2+\xi^2)}{4y}\, d\xi \tag{2.4}$$

It is readily verified that for $y > 0$ and any x the function $K(x, y)$ satisfies Equation (2.2). We will show that (2.3) is likewise satisfied. Since we are interested in the values of $K(x, y)$ for small y, the Bessel function under the integral sign may be replaced by its asymptotic representation

$$I_0(z) \approx \frac{1}{\sqrt{2\pi z}} e^z$$

Then, introducing the new integration variable

$$\alpha = \frac{x-\xi}{2\sqrt{y}}$$

we will have for $\delta_0 < x < 1$

$$\lim_{y\to+0} K(x, y) = \frac{1}{\sqrt{\pi x}} \lim_{y\to+0} \int_{\theta_1}^{\theta_2} \sqrt{x + 2\alpha\sqrt{y}}\,[S(x + 2\alpha\sqrt{y}) +$$

$$+\Phi(x+2\alpha\sqrt{y})]\,e^{-\alpha^2}d\alpha = Sx+\Phi(x) \qquad \left(\theta_1=\frac{\delta_0-x}{2\sqrt{y}},\quad \theta_2=\frac{1-x}{2\sqrt{y}}\right) \tag{2.5}$$

The justification of the limit process under the integral sign follows from the continuity of $\Phi(x)$.

At the ends of the interval the limit depends on the method of approach to the points $M_1(1,\ 0)$ and $M_2(\delta_0,\ 0)$. For passage along straight lines $x = 1$, $x = \delta_0$ it is equal to $1/2\ S$ and $1/2\ [S\,\delta_0 + \Phi(\delta_0)]$, respectively. For the function $\lambda(x,\ y)$ we will have the following boundary problem:

$$\frac{\partial\lambda}{\partial y}=\frac{\partial^2\lambda}{\partial x^2}+\frac{1}{x}\ \frac{\partial\lambda}{\partial x} \qquad \text{for } y>0,\ \delta(y)<x<1 \tag{2.6}$$

$$\lim_{y\to+0}\lambda(x,y)=0 \qquad \text{for } \delta_0<x<1 \tag{2.7}$$

$$\lim_{x\to1-0}\lambda(x,y)=S-\int_0^y P_*(\sigma)\,d\sigma- \tag{2.8}$$

$$-\frac{1}{2y}\int_{\delta_0}^1\xi\,[S\xi+\Phi(\xi)]\,I_0\left(\frac{x\zeta}{2y}\right)\exp\frac{-(1+\xi^2)}{4y}\,d\xi=f(y)$$

$$\lim_{x\to\delta(y)+0}\lambda(x,y)=\Phi(\delta_0)+S\delta(y)-2S\int_0^y\frac{d\sigma}{\delta(\sigma)}- \qquad (y>0) \tag{2.9}$$

$$-\frac{1}{2y}\int_{\delta_0}^1\xi\,[S\xi+\Phi(\xi)]\,I_0\left(\frac{\xi\delta(y)}{2y}\right)\exp\frac{-[\xi^2+\delta^2(y)]}{4y}\,d\xi=\varphi(y)$$

$$\lim_{x\to\delta(y)+0}\frac{\partial\lambda}{\partial r}=S-\frac{1}{4y^2}\int_{\delta_0}^1\xi\,[S\xi+\Phi(\xi)]\left\{\xi I_1\left(\frac{\xi\delta(y)}{2y}\right)-\right. \qquad (y>0) \tag{2.10}$$

$$\left.-\delta(y)\,I_0\left(\frac{\xi\delta(y)}{2y}\right)\right\}\exp\frac{-[\xi^2+\delta^2(y)]}{4y}\,d\xi=\psi(y)$$

We will seek the solution of the problem (2.6)-(2.10) in the form of the sum of a regular solution $\theta(x,\ y)$ of (2.6) which satisfies the conditions

$$\theta(x,0)=0,\qquad \theta(1,y)=f(y)-N(1,y),\qquad |\theta(0,y)|\leqslant C \tag{2.11}$$

and a singular solution $N(x,\ y)$ which, except for the zero initial condition, satisfies also two discontinuity conditions on the arbitrary curve $x = \delta(y)$

$$\lim_{x\to\delta(y)+0} N(x,y) - \lim_{x\to\delta(y)-0} N(x,y) = \varphi(y) \qquad \lim_{x\to\delta(y)+0} \frac{\partial N}{\partial x} - \lim_{x\to\delta(y)-0} \frac{\partial N}{\partial x} = \psi(y) \tag{2.12}$$

and the conditions

$$|N(0,y)| \leqslant A, \qquad N(\infty, y) = 0 \tag{2.13}$$

The regular solution should exist in the region $T\{0 \leqslant y \leqslant y_0,\ 0 \leqslant x < 1\}$. As regards the singular solution, we assume its existence in the following. We will denote by D_+ the region $\{0 < y \leqslant y_0,\ \delta(y) < x < \infty\}$, by D_- the region $\{0 < y \leqslant y_0,\ 0 \leqslant x < \delta(y)\}$, complementary to D_+. Also let $\bar{D}$ be the closure $D_- + D_+$ in the manifold $E\{0 \leqslant y \leqslant y_0,\ 0 \leqslant x < \infty,\ |x - \delta_0| + |y| > 0\}$ and D the interior of $\bar{D}$. Obviously neither D nor $\bar{D}$ depend on the choice of $\delta(y)$. Below we will show that there exists a unique bounded solution of the boundary value problem (2.12)-(2.13) in the region D.

Construction of the regular solution. From a physical point of view the regular solution may be understood as a distribution of temperature in the infinite rod with a conductivity equal to unity, having initially zero temperature and its surface being maintained at the temperature

$$\alpha(y) = f(y) - N(1, y)$$

It is known that such a solution may be written in the form

$$\theta(x,y) = \alpha(y) - 2\alpha(0) \sum_{k=1}^{\infty} \frac{J_0(\alpha_k x)}{\alpha_k J_1(\alpha_k)} \exp(-\alpha_k^2 y) -$$
$$- 2\sum_{k=1}^{\infty} \frac{J_0(\alpha_k x)}{\alpha_k J_1(\alpha_k)} \exp(-\alpha_k^2 y) \int_0^y \alpha'(\sigma) \exp(\alpha_k^2 \sigma)\, d\sigma \tag{2.14}$$

where α_k is the root of the equation $J_0(\alpha) = 0$, and $J_0(\alpha)$ and $J_1(\alpha)$ are Bessel functions of zero and first order.

Construction of the singular solution. The construction of the singular solution will be based on properties of plane thermal potentials of the simple and double layers of sources distributed uniformly over the circle. We will show that the singular solution may be represented in the form of a combination of thermal potentials so that all boundary conditions will be fulfilled. Let

$$N(x,y) = K_1(x,y) + K_2(x,y) \tag{2.15}$$

where

$$K_1(x,y) = \frac{1}{4} \int_0^y \frac{\varphi(\eta)\,\delta(\eta)}{(y-\eta)^2} \left\{ x I_1\!\left(\frac{x\delta(\eta)}{2(y-\eta)}\right) - \delta(\eta)\, I_0\!\left(\frac{x\delta(\eta)}{2(y-\eta)}\right) \right\} \times$$

$$\times \exp\frac{-[x^2+\delta^2(\eta)]}{4(y-\eta)}\,d\eta - \frac{1}{2}\int\limits_0^y \frac{\varphi(\eta)\,\delta(\eta)\,\delta'(\eta)}{y-\eta}\times$$

$$\times I_0\left(\frac{x\delta(\eta)}{2(y-\eta)}\right)\exp\frac{-[x^2+\delta^2(\eta)]}{4(y-\eta)}\,d\eta. \qquad (2.16)$$

$$K_2(x,y) = -\frac{1}{4}\int\limits_0^y \frac{2\psi(\eta)\,\delta(\eta)}{y-\eta}\,I_0\left(\frac{x\delta(\eta)}{2(y-\eta)}\right)\exp\frac{-[x^2+\delta^2(\eta)]}{4(y-\eta)}\,d\eta \qquad (2.17)$$

It is readily verified that $K_1(x, y)$ satisfies Equation (2.6) for $x \neq \delta(y)$, the zero initial condition and the conditions for $x = 0$ and $x \to \infty$. For this purpose it will be assumed that the function $\delta(y)$ is continuously differentiable, that it vanishes nowhere and that its derivative $\delta'(y) < c/\sqrt{y}$. The function $\Phi(x)$ is continuous, differentiable and $\Phi'(x)$ satisfies a Lipschitz condition in the interval $\delta_0 < x < 1$. We will show that $K_1(x, y)$ has on the curve $x = \delta(y)$ discontinuities equal to $\phi(y)$ and that its derivative $\partial K_1(x, y)/\partial x$ has discontinuities $-1/2\,\phi(y)/\delta(y)$.

We will divide the integration interval into two parts: from 0 to $y - \epsilon$ and from $y - \epsilon$ to y; then, replacing in the interval $(y - \epsilon < \eta < y)$ the Bessel function by its asymptotic expansion

$$I_\eta(z) \sim \frac{1}{\sqrt{2\pi z}}\,e^z$$

we will have

$$K_1(x,y) = \frac{1}{4}\int\limits_0^{y-\varepsilon} \frac{\varphi(\eta)\,\delta(\eta)}{(y-\eta)^2}\left\{yI_1\left(\frac{x\delta(\eta)}{2(y-\eta)}\right) - \delta(\eta)\,I_0\left(\frac{x\delta(\eta)}{2(y-\eta)}\right)\right\}\times$$

$$\times \exp\frac{-[x^2+\delta^2(\eta)]}{4(y-\eta)}\,d\eta - \frac{1}{2}\int\limits_0^{y-\varepsilon} \frac{\varphi(\eta)\,\delta(\eta)\,\delta'(\eta)}{y-\eta}\,I_0\left(\frac{x\delta(\eta)}{2(y-\eta)}\right)\times$$

$$\times \exp\frac{-[x^2+\delta^2(\eta)]}{4(y-\eta)}\,d\eta + \frac{1}{\sqrt{\pi x}}\int\limits_{y-\varepsilon}^{y} \varphi(\eta)\sqrt{\delta(\eta)}\times$$

$$\times\left\{\frac{x-\delta(\eta)}{4(y-\eta)^{3/2}} - \frac{\delta'(\eta)}{2(y-\eta)^{1/2}}\right\}\exp\frac{-[x-\delta(\eta)]^2}{4(y-\eta)}\,d\eta \qquad (2.18)$$

The first two terms have no singularities and are continuously differentiable functions, the third term, apart from a factor, represents the sum of linear thermal potentials of a simple and a double layer for which it has been shown [1,2] that it has discontinuities $\phi(y)$. Differentiating (2.18) with respect to x and using a property of linear thermal potentials, it may be shown that $\partial K_1/\partial x$ has on the curve $x = \delta(y)$ the discontinuities

$$\lim_{x\to\delta(y)+0}\frac{\partial k_1}{\partial x}-\lim_{x\to\delta(y)-0}\frac{\partial k_1}{\partial x}=-\frac{1}{2}\frac{\varphi(y)}{\delta(y)}$$

Performing the same transformation as above we will express $K_2(x, y)$ by a linear thermal potential of a simple layer. Hence, the function $K_2(x, y)$ will be continuous and its derivative will have on the curve $x = \delta(y)$ the discontinuities

$$\lim_{x\to\delta(y)+0}\frac{\partial k_2}{\partial x}-\lim_{x\to\delta(y)-0}\frac{\partial k_2}{\partial x}=\psi(y)+\frac{1}{2}\frac{\varphi(y)}{\delta(y)}$$

Thus, the singular solution satisfies all the imposed conditions. The uniqueness proof may be carried out in the same manner as in [1,2].

The sum of the regular and of the singular solutions will, in addition to the conditions (2.12), satisfy the condition (2.8), being the solution of (2.6) in the region $\{0 < y < y_0, 0 < x < 1\}$. But this solution still contains a derivative of $\delta(y)$. We will demand that

$$\lim_{x\to\delta(y)-0}\lambda(x, y)=0, \qquad \lim_{x\to\delta(y)-0}\frac{\partial\lambda(x, y)}{\partial x}=0 \tag{2.19}$$

Then $\lambda(x, y)$ in the region $\{0 < y < y_0, \delta(y) < x < 1\}$ will represent the unknown solution of the problem (2.6)-(2.10). The conditions (2.19) may be considered as equations for the determination of the function $\delta(y)$. It may be shown that any solution of the first equation (2.19) satisfies simultaneously the second and vice-versa. Thus, if one of the equations (2.19) had a unique solution, then this solution gives also the law of change of the boundary of the "kernel" with time, and (2.1) the distribution of velocities of the visco-plastic flow.

In conclusion, we will note that an analogous problem has been studied by Krasil'nikov [3]. In our opinion, his solution may not be considered to be exact, since the author's formulation of the problem raises some objections. The author assumes from the start that the visco-plastic zone of flow occupies the entire cross-section of the tube and seeks the solution of the equation of motion, satisfying the initial condition and the condition of attachment at the wall; then, based on the well-known fact that the plastic deformation may not spread to the axis of the tube, he assumes the existence of the kernel of the flow, the boundary of which he finds from the condition that no slip can occur on the boundary. From the physical point of view such a formulation of the problem means a denial of the influence of the boundary of the kernel on the development of the flow, which naturally affects the constructed solution (the last does not enter into it). For the correct formulation of the problem the region of existence of the solution is necessarily assumed to be the region between the wall of the tube and the unknown moving kernel, at the boundary of which two conditions must be given.

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